8.1 Complex Numbers

Question Paper

Course	CIE A Level Maths
Section	8. Complex Numbers
Topic	8.1 Complex Numbers
Difficulty	Very Hard

Time allowed: 60

Score: /49

Percentage: /100

Question 1

For a general complex number $z = x + \mathrm{i}y$, where $x, y \in \mathbb{R}$ and $z \neq 0$, show that

(i) $\mathrm{Re}\left(\frac{1}{z} + \frac{1}{z^*}\right) = \frac{2x}{x^2+y^2}$

(ii) $\mathrm{Im}\left(\frac{1}{z} + \frac{1}{z^*}\right) = 0$

[3 marks]

Question 2

Given that $z_1 = 3 - 4\mathrm{i}$, $z_2 = a + b\mathrm{i}$, and $\frac{2z_1}{z_2} = -1 - 7\mathrm{i}$, where a and b are real numbers, find the values of a and b.

[3 marks]

Question 3

Find all the complex numbers z for which $z^2 = 4z^*$.

[4 marks]

Question 4

α, β and γ are the three roots of the cubic equation

$$z^3 + (b-3)z^2 + (7-3b)z - 21 = 0$$

where $b \in \mathbb{R}$.

Given that $\alpha = 3$ and that β and γ are distinct non-real complex numbers, find the range of possible values of b.

[4 marks]

Question 5

Given that $3 + q\text{i}$ is one of the roots of the quadratic equation $z^2 - 12pz + 58 = 0$, where p and q are positive real constants, find the values of p and q.

[4 marks]

Question 6

Work out the solutions to the equation $z^2 - 6\mathrm{i}z - 14 = 0$. Be sure to show clear algebraic working.

[4 marks]

Question 7a

You are given that the complex number $\alpha = 1 - 3\mathrm{i}$ satisfies the cubic equation

$$z^3 + 4z^2 + kz + m = 0,$$

where k and m are real constants.

(a) By first calculating α^2 and α^3, find the values of k and m.

[5 marks]

Question 7b

(b) Find the other two roots of the cubic equation.

[3 marks]

Question 8a

$f(z) = z^4 - 2z^3 + 6z^2 + pz + 125$, where p is a real constant.

(a) Given that $3 - 4i$ is a root of the equation $f(z) = 0$, show that $z^2 + 4z + 5$ is a factor of $f(z)$.

[4 marks]

Question 8b

(b) Hence find the value of p and solve completely the equation $\mathrm{f}(z) = 0$.

[3 marks]

Question 9a

The principal square root of a complex number z is defined as

$$\sqrt{z} = x + \mathrm{i}y$$

where x and y are real numbers and $x \geq 0$. If $x = 0$ then the value for y is chosen such that $y \geq 0$. Note that the other square root of z will then be given by $-\sqrt{z} = -x - \mathrm{i}y$.

(a) Show that

$$x = \sqrt{\frac{\mathrm{Re}(z) + \sqrt{\left(\mathrm{Re}(z)\right)^2 + \left(\mathrm{Im}(z)\right)^2}}{2}}$$

[5 marks]

Question 9b

(b) Given that $x > 0$, derive a formula for y in terms of x and $\text{Im}(z)$, and explain why y in this case will always have the same sign (positive, negative, or zero) as $\text{Im}(z)$.

[2 marks]

Question 9c

(c) Hence show that in general

$$y = \pm\sqrt{\frac{-\text{Re}(z) + \sqrt{\left(\text{Re}(z)\right)^2 + \left(\text{Im}(z)\right)^2}}{2}}$$

with the choice of the positive or negative value being dependent on the properties of z.

[2 marks]

Question 9d

(d) Explain what must be true of z for each of the following to be true:

(i) $x = 0, y \neq 0$

(ii) $x \neq 0, y = 0$

(iii) $x = 0, y = 0$

[3 marks]